

**SEMICLASSICAL ASYMPTOTICS
AND GAPS IN THE SPECTRA
OF MAGNETIC SCHRÖDINGER OPERATORS**

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ABSTRACT. In this paper, we study an L^2 version of the semiclassical approximation of magnetic Schrödinger operators with invariant Morse type potentials on covering spaces of compact manifolds. In particular, we are able to establish the existence of an arbitrary large number of gaps in the spectrum of these operators, in the semiclassical limit as the coupling constant μ goes to zero.

INTRODUCTION

A charged particle constrained to a manifold, moves along geodesic orbits. However, in the presence of a magnetic field, the orbit of the charged particle is no longer a geodesic, but rather a *magnetic geodesic*, that is, a trajectory determined by solving the Hamiltonian equations given by the symbol of the magnetic Hamiltonian. In the case of Euclidean space and constant magnetic field, the magnetic geodesics are circles, and on the hyperbolic space they are the lines of tangency of the hyperboloids embedded in the Minkowski space. The corresponding magnetic Schrödinger operators are sometimes also called *magnetic Hamiltonians* or Landau Hamiltonians. The magnetic Schrödinger operators turn out to possess rich spectral properties cf. [1], [4], [8], [9], [16], that are important in the analysis of the quantum Hall effect cf. [3], [6], [15], [21].

We begin by reviewing the construction of the magnetic Hamiltonian on the universal covering space \widetilde{M} of a compact connected manifold M . The fundamental group $\Gamma = \pi_1(M)$ acts on \widetilde{M} by the deck transformations, so that $\widetilde{M}/\Gamma = M$. Let ω be a closed 2-form on M and B be its lift to \widetilde{M} , so that B is a Γ -invariant closed 2-form on \widetilde{M} . We assume that B is exact. Pick a 1-form A on \widetilde{M} such that $dA = B$. As in geometric quantization we may regard A as defining a Hermitian connection $\nabla = d + iA$ on the trivial line bundle \mathcal{L} over \widetilde{M} , whose curvature is iB . Physically we can think of A as the electromagnetic vector potential for a magnetic field B . Using the Riemannian metric the magnetic Laplacian is given by

$$H_A = \nabla^* \nabla = (d + iA)^*(d + iA).$$

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Then the Schrödinger equation describing the quantum mechanics of a single electron of mass m which is confined to move on the covering space \widetilde{M} in the presence of a periodic magnetic field is given by

$$i\hbar \frac{\partial}{\partial t} \psi = \frac{1}{2m} (\hbar d + ieA)^* (\hbar d + ieA) \psi + \mu^{-2} V \psi$$

where H_A is the magnetic Laplacian, V is a Γ -invariant electric potential function, \hbar is Planck's constant, e is the electric charge of the charge carrier and μ is the coupling constant. In the time independent framework, the relevant operator is the magnetic Schrödinger operator

$$H_{A,V}(\mu) = \mu H_A + \mu^{-1} V,$$

where the physical constants are set equal to 1. It is the qualitative aspects of the spectrum of $H_{A,V}(\mu)$ that are relevant to the study of the quantum Hall effect. An important feature of the magnetic Schrödinger operator is that it commutes with magnetic translations, that is with a projective action of the fundamental group Γ . Let σ denote the multiplier or $U(1)$ -valued 2-cocycle on Γ defining this projective action. Under the assumption that the Kadison constant of $C_r^*(\Gamma, \sigma)$ is positive, it was proved by Brüning and Sunada [4] that there are only a finite number of gaps in the spectrum of the magnetic Schrödinger operator $H_{A,V}$ that lie in any left half-line $(-\infty, \lambda]$. In fact, under the same hypotheses, they obtain Weyl-type asymptotics of the number of gaps in the spectrum, as $\lambda \rightarrow \infty$. On the other hand, it was proved in [16] that the Kadison constant is positive under the assumptions that the Baum-Connes conjecture with coefficients holds for Γ , and that Γ has finite cohomological dimension and finally that σ defines a rational cohomology class. The assumptions on Γ are satisfied for instance when Γ is a discrete subgroup of $SO(1, n)$ or $SU(1, n)$ or of an amenable Lie group. Our paper on the other hand proves that $H_{A,V}$ can have arbitrarily large number of gaps, if V is chosen to be a suitable Morse-type potential. An outstanding open problem is to construct magnetic Schrödinger operators $H_{A,V}$ that have infinitely many gaps in some half-line $(-\infty, \lambda]$, for any Γ , where of course σ has to be an irrational cohomology class. This has been shown to be the case at least in the discretization of the magnetic Schrödinger operator, called the Harper operator, and when $\Gamma = \mathbb{Z}^2$, cf. [2], [7], [13]. There have been other interesting papers on gaps in the spectrum of magnetic Schrödinger operators on covering spaces. One such is [14] where the asymptotics of the size of gaps in the spectrum are obtained on hyperbolic spaces, as the mid-point of the gap tends to infinity, generalizing known results on Euclidean spaces. In two dimensions, magnetic Schrödinger operators on covering spaces are the Hamiltonians in the model for the integer quantum Hall effect when the covering space is the Euclidean plane [3], and for the fractional quantum Hall effect when the covering space is the hyperbolic plane [15], [6]. The standard assumption made is that the Fermi energy level lies in a gap of the spectrum of the Hamiltonian (which however can be relaxed to the assumption that it lies in a gap in extended states of the Hamiltonian after further analysis).

In this paper, we study the semiclassical approximation, as the coupling constant μ approaches zero, of the Hamiltonians $H_{A,V}(\mu)$, for a class of *Morse type* potentials V , which include all functions $V = |df|^2$, where f is a Γ -invariant Morse function on \widetilde{M} , that is f is the lift to \widetilde{M} of a Morse function on M . We show that the spectrum of this operator, is approximated by the union of the spectra of model operators which are defined near the critical points of the Morse potential V . In particular, we are able to deduce the existence of an arbitrarily large number of gaps in the spectrum of these Hamiltonians, for μ sufficiently small, a fact which is of crucial importance in

the study of the quantum Hall effect. We adapt the L^2 version of semiclassical approximation of Witten [20], [10], [19] to our context.

The paper is organized as follows. We first give a summary of our main results. Then we recall some preliminary material on projective unitary representations and the associated von Neumann algebra of operators commuting with this algebra, together with the von Neumann trace and von Neumann dimension function. In the next section, we establish the first main theorem on the existence of spectral gaps for large values of the coupling parameter or equivalently for small values of the coupling constant μ . Here we use the method of semiclassical approximation.

Summary of main results. In this paper, we will assume that the potential V is a smooth Γ -invariant function which satisfies the following *Morse type condition*: $V(x) \geq 0$ for all $x \in \widetilde{M}$. Also if $V(x_0) = 0$ for some x_0 in \widetilde{M} , then there is a positive constant c such that $V(x) \geq c|x - x_0|^2$ for all x in a neighborhood of x_0 . We will also assume that V has at least one zero point.

We remark that all functions $V = |df|^2$, where $|df|$ denotes the pointwise norm of the differential of a Γ -invariant Morse function f on \widetilde{M} , are examples of Morse type potentials.

We next enunciate the principle on which this paper is based. Associated to each Hamiltonian $H_{A,V}(\mu)$, there is a *model operator* K (cf. section 2) which has a discrete spectrum. It is defined as a direct sum of harmonic oscillators, associated with the potential wells of V in a fundamental domain of Γ in \widetilde{M} .

Semiclassical approximation principle: *Let V be a Morse type potential. Then in the semiclassical limit as the coupling constant μ goes to zero, the spectrum of the Hamiltonian $H_{A,V}(\mu)$ “tears up” into bands which are located near the eigenvalues of the associated model operator K .*

The following main theorem establishes the existence of arbitrarily large number of gaps in the spectrum of $H_{A,V}(\mu)$ whenever μ is sufficiently small. The proof of this theorem uses an analogue of Witten’s semiclassical approximation technique for proving the Morse inequalities. The L^2 -analogue (in the absence of a magnetic field) was proved by Shubin [19], also Burghhelea et al. [5]. We modify the proof in [19] to obtain the result in section 2. The physical explanation for the appearance of gaps in the spectrum $H_{A,V}(\mu)$ is that the potential wells get deeper as $\mu \rightarrow 0$ and the atoms get (asymptotically) isolated, so that the energy levels of $H_{A,V}(\mu)$ are approximated by those of the corresponding model operator K .

Theorem (Existence of spectral gaps). *Let V be a Morse type potential. If $E \in \mathbb{C}$ is such that $E \notin \text{spec}(K)$, then there exists $\mu_0 > 0$ such that for all $\mu \in (0, \mu_0)$, E is in the resolvent set of $H_{A,V}(\mu)$. If in addition E is real and lies between two eigenvalues of K , then E is in a spectral gap of $H_{A,V}(\mu)$.*

Since the spacing between the eigenvalues of K is bounded below, it follows that there exists arbitrarily large number of gaps in the spectrum of $H_{A,V}(\mu)$ whenever the coupling constant μ is sufficiently small.

Remark. We observe that the operator $H_A + \mu^{-2}V = \mu^{-1}H_{A,V}(\mu)$ also has arbitrarily large number of gaps in its spectrum whenever the coupling constant μ is sufficiently small. Analogous results in the special case of Euclidean space were obtained in [17].

1. PRELIMINARIES

Let M be a compact connected Riemannian manifold, Γ be its fundamental group and \widetilde{M} be its universal cover, i.e. one has the principal bundle $\Gamma \rightarrow \widetilde{M} \xrightarrow{p} M$. To make the paper self-contained, we include preliminary material, some of which may not be new, cf. [1], [4], [6], [15], [16].

1.1. Projective action, or magnetic translations. Let ω be a closed real-valued 2-form on M such that $B = p^*\omega$ is *exact*. So $B = dA$ where A is a 1-form on \widetilde{M} . We will assume A without loss of generality that A is real-valued too. Define $\nabla = d + iA$. Then ∇ is a Hermitian connection on the trivial line bundle over \widetilde{M} with the curvature $(\nabla)^2 = iB$. The connection ∇ defines a projective action of Γ on L^2 functions as follows.

Observe that since B is Γ -invariant, one has $0 = \gamma^*B - B = d(\gamma^*A - A) \quad \forall \gamma \in \Gamma$. So $\gamma^*A - A$ is a closed 1-form on the simply connected manifold \widetilde{M} , therefore

$$\gamma^*A - A = d\psi_\gamma, \quad \forall \gamma \in \Gamma,$$

where ψ_γ is a smooth function on \widetilde{M} . It is defined up to an additive constant, so we can assume in addition that it satisfies the following normalization condition:

- $\psi_\gamma(x_0) = 0$ for a fixed $x_0 \in \widetilde{M}, \quad \forall \gamma \in \Gamma$.

It follows that ψ_γ is real-valued and $\psi_e(x) \equiv 0$, where e denotes the neutral element of Γ . It is also easy to check that

- $\psi_\gamma(x) + \psi_{\gamma'}(\gamma x) - \psi_{\gamma'\gamma}(x)$ is independent of $x \in \widetilde{M}, \quad \forall \gamma, \gamma' \in \Gamma$.

Then $\sigma(\gamma, \gamma') = \exp(-i\psi_\gamma(\gamma' \cdot x_0))$ defines a *multiplier* on Γ i.e. $\sigma : \Gamma \times \Gamma \rightarrow U(1)$ satisfies

- $\sigma(\gamma, e) = \sigma(e, \gamma) = 1, \quad \forall \gamma \in \Gamma;$
- $\sigma(\gamma_1, \gamma_2)\sigma(\gamma_1\gamma_2, \gamma_3) = \sigma(\gamma_1, \gamma_2\gamma_3)\sigma(\gamma_2, \gamma_3), \quad \forall \gamma_1, \gamma_2, \gamma_3 \in \Gamma \quad (\text{the cocycle relation}).$

It follows from these relations that $\sigma(\gamma, \gamma^{-1}) = \sigma(\gamma^{-1}, \gamma)$.

The complex conjugate multiplier $\bar{\sigma}(\gamma, \gamma') = \exp(i\psi_\gamma(\gamma' \cdot x_0))$ also satisfies the same relations.

For $u \in L^2(\widetilde{M})$ and $\gamma \in \Gamma$ define

$$U_\gamma u = (\gamma^{-1})^*u, \quad S_\gamma u = \exp(-i\psi_\gamma)u.$$

Then the operators $T_\gamma = U_\gamma \circ S_\gamma$ satisfy

$$T_e = \text{Id}, \quad T_{\gamma_1} T_{\gamma_2} = \sigma(\gamma_1, \gamma_2) T_{\gamma_1\gamma_2},$$

for all $\gamma_1, \gamma_2 \in \Gamma$. In this case one says that the map $T : \Gamma \rightarrow \mathcal{U}(L^2(\widetilde{M}))$, $\gamma \mapsto T_\gamma$, is a projective (Γ, σ) -unitary representation, where for any Hilbert space \mathcal{H} we denote by $\mathcal{U}(\mathcal{H})$ the group of all unitary operators in \mathcal{H} . In other words one says that the map $\gamma \mapsto T_\gamma$ defines a (Γ, σ) -action in \mathcal{H} .

It is also easy to check that the adjoint operator to T_γ in $L^2(\widetilde{M})$ (with respect to a smooth Γ -invariant measure) is

$$T_\gamma^* = \bar{\sigma}(\gamma, \gamma^{-1}) T_{\gamma^{-1}}.$$

The operators T_γ are also called *magnetic translations*.

1.2. Twisted group algebras. Denote by $\ell^2(\Gamma)$ the standard Hilbert space of complex-valued L^2 -functions on the discrete group Γ . We will use a left $(\Gamma, \bar{\sigma})$ -action on $\ell^2(\Gamma)$ (or, equivalently, a $(\Gamma, \bar{\sigma})$ -unitary representation in $\ell^2(\Gamma)$) which is given explicitly by

$$T_\gamma^L f(\gamma') = f(\gamma^{-1}\gamma') \bar{\sigma}(\gamma, \gamma^{-1}\gamma'), \quad \gamma, \gamma' \in \Gamma.$$

It is easy to see that this is indeed a $(\Gamma, \bar{\sigma})$ -action, i.e.

$$T_e^L = \text{Id} \quad \text{and} \quad T_{\gamma_1}^L T_{\gamma_2}^L = \bar{\sigma}(\gamma_1, \gamma_2) T_{\gamma_1 \gamma_2}^L, \quad \forall \gamma_1, \gamma_2 \in \Gamma.$$

Also

$$(T_\gamma^L)^* = \sigma(\gamma, \gamma^{-1}) T_{\gamma^{-1}}^L.$$

Let

$$\mathcal{A}^R(\Gamma, \sigma) = \left\{ A \in \mathcal{B}(\ell^2(\Gamma)) : [T_\gamma^L, A] = 0, \quad \forall \gamma \in \Gamma \right\}$$

be the commutant of the left $(\Gamma, \bar{\sigma})$ -action on $\ell^2(\Gamma)$. Here by $\mathcal{B}(\mathcal{H})$ we denote the algebra of all bounded linear operators in a Hilbert space \mathcal{H} . By the general theory, $\mathcal{A}^R(\Gamma, \sigma)$ is a von Neumann algebra and is known as the *(right) twisted group von Neumann algebra*. It can also be realized as follows. Let us define the following operators in $\ell^2(\Gamma)$:

$$T_\gamma^R f(\gamma') = f(\gamma'\gamma) \sigma(\gamma', \gamma), \quad \gamma, \gamma' \in \Gamma.$$

It is easy to check that they form a right (Γ, σ) -action in $\ell^2(\Gamma)$ i.e.

$$T_e^R = \text{Id} \quad \text{and} \quad T_{\gamma_1}^R T_{\gamma_2}^R = \sigma(\gamma_1, \gamma_2) T_{\gamma_1 \gamma_2}^R, \quad \forall \gamma_1, \gamma_2 \in \Gamma,$$

and also

$$(T_\gamma^R)^* = \bar{\sigma}(\gamma, \gamma^{-1}) T_{\gamma^{-1}}^R.$$

This action commutes with the left $(\Gamma, \bar{\sigma})$ -action defined above i.e.

$$T_\gamma^L T_{\gamma'}^R = T_{\gamma'}^R T_\gamma^L, \quad \forall \gamma, \gamma' \in \Gamma.$$

It can be shown that the von Neumann algebra $\mathcal{A}^R(\Gamma, \sigma)$ is generated by the operators $\{T_\gamma^R\}_{\gamma \in \Gamma}$ (see e.g. a similar argument in [18]).

Similarly we can introduce a von Neumann algebra

$$\mathcal{A}^L(\Gamma, \bar{\sigma}) = \left\{ A \in \mathcal{B}(\ell^2(\Gamma)) : [T_\gamma^R, A] = 0, \quad \forall \gamma \in \Gamma \right\}.$$

We will refer to it as *(left) twisted group von Neumann algebra*. It is generated by the operators $\{T_\gamma^L\}_{\gamma \in \Gamma}$, and it is the commutant of $\mathcal{A}^R(\Gamma, \sigma)$.

Let us define a twisted group algebra $\mathbb{C}(\Gamma, \sigma)$ which consists of complex valued functions with finite support on Γ and with the twisted convolution operation

$$(f * g)(\gamma) = \sum_{\gamma_1, \gamma_2 : \gamma_1 \gamma_2 = \gamma} f(\gamma_1)g(\gamma_2)\sigma(\gamma_1, \gamma_2).$$

The basis of $\mathbb{C}(\Gamma, \sigma)$ as a vector space is formed by δ -functions $\{\delta_\gamma\}_{\gamma \in \Gamma}$, $\delta_\gamma(\gamma') = 1$ if $\gamma = \gamma'$ and 0 otherwise. We have

$$\delta_{\gamma_1} * \delta_{\gamma_2} = \sigma(\gamma_1, \gamma_2)\delta_{\gamma_1 \gamma_2}.$$

Associativity of this multiplication is equivalent to the cocycle condition.

Note also that the δ -functions $\{\delta_\gamma\}_{\gamma \in \Gamma}$ form an orthonormal basis in $\ell^2(\Gamma)$. It is easy to check that

$$T_\gamma^L \delta_{\gamma'} = \bar{\sigma}(\gamma, \gamma')\delta_{\gamma\gamma'}, \quad T_\gamma^R \delta_{\gamma'} = \sigma(\gamma'\gamma^{-1}, \gamma)\delta_{\gamma'\gamma-1}.$$

It is clear that the correspondences $\delta_\gamma \mapsto T_\gamma^L$ and $\delta_\gamma \mapsto T_\gamma^R$ define representations of $\mathbb{C}(\Gamma, \bar{\sigma})$ and $\mathbb{C}(\Gamma, \sigma)$ respectively. In both cases the weak closure of the image of the twisted group algebra coincides with the corresponding von Neumann algebra $(\mathcal{A}^L(\Gamma, \bar{\sigma}))$ and $(\mathcal{A}^R(\Gamma, \sigma))$ respectively). The corresponding norm closures are so called *reduced twisted group C*-algebras* which are denoted $C_r^*(\Gamma, \bar{\sigma})$ and $C_r^*(\Gamma, \sigma)$ respectively.

The von Neumann algebras $\mathcal{A}^L(\Gamma, \bar{\sigma})$ and $\mathcal{A}^R(\Gamma, \sigma)$ can be described in terms of the matrix elements. For any $A \in \mathcal{B}(\ell^2(\Gamma))$ denote $A_{\alpha, \beta} = (A\delta_\beta, \delta_\alpha)$ (which is a matrix element of A). Then repeating standard arguments (given in a similar situation e.g. in [18]) we can prove that for any $A \in \mathcal{B}(\ell^2(\Gamma))$ the inclusion $A \in \mathcal{A}^R(\Gamma, \sigma)$ is equivalent to the relations

$$A_{\gamma x, \gamma y} = \bar{\sigma}(\gamma, x)\sigma(\gamma, y)A_{x, y}, \quad \forall x, y, \gamma \in \Gamma.$$

In particular, we have for any $A \in \mathcal{A}^R(\Gamma, \sigma)$

$$A_{\gamma x, \gamma x} = A_{x, x}, \quad \forall x, \gamma \in \Gamma.$$

Similarly, for any $A \in \mathcal{B}(\ell^2(\Gamma))$ the inclusion $A \in \mathcal{A}^L(\Gamma, \bar{\sigma})$ is equivalent to the relations

$$A_{x\gamma, y\gamma} = \bar{\sigma}(x, \gamma)\sigma(y, \gamma)A_{x, y}, \quad \forall x, y, \gamma \in \Gamma.$$

In particular, we have

$$A_{x\gamma, x\gamma} = A_{x, x}, \quad \forall x, \gamma \in \Gamma,$$

for any $A \in \mathcal{A}^L(\Gamma, \bar{\sigma})$.

A finite von Neumann trace $\text{tr}_{\Gamma, \bar{\sigma}} : \mathcal{A}^L(\Gamma, \bar{\sigma}) \rightarrow \mathbb{C}$ is defined by the formula

$$\text{tr}_{\Gamma, \bar{\sigma}} A = (A\delta_e, \delta_e).$$

We can also write $\text{tr}_{\Gamma, \bar{\sigma}} A = A_{\gamma, \gamma} = (A\delta_\gamma, \delta_\gamma)$ for any $\gamma \in \Gamma$ because the right hand side does not depend of γ .

A finite von Neumann trace $\text{tr}_{\Gamma, \sigma} : \mathcal{A}^R(\Gamma, \sigma) \rightarrow \mathbb{C}$ is defined by the same formula, so we will denote by tr_Γ any of these traces.

Let \mathcal{H} denote an infinite dimensional complex Hilbert space. Then the Hilbert tensor product $\ell^2(\Gamma) \otimes \mathcal{H}$ is both $(\Gamma, \bar{\sigma})$ -module and (Γ, σ) -module under the actions $\gamma \mapsto T_\gamma^L \otimes 1$ and $\gamma \mapsto T_\gamma^R \otimes 1$

respectively. Let $\mathcal{A}_{\mathcal{H}}^L(\Gamma, \bar{\sigma})$ and $\mathcal{A}_{\mathcal{H}}^R(\Gamma, \sigma)$ denote the von Neumann algebras in $\ell^2(\Gamma) \otimes \mathcal{H}$ which are commutants of the (Γ, σ) - and $(\Gamma, \bar{\sigma})$ -actions respectively. Clearly $\mathcal{A}_{\mathcal{H}}^L(\Gamma, \bar{\sigma}) \cong \mathcal{A}^L(\Gamma, \bar{\sigma}) \otimes \mathcal{B}(\mathcal{H})$ and $\mathcal{A}_{\mathcal{H}}^R(\Gamma, \sigma) \cong \mathcal{A}^R(\Gamma, \sigma) \otimes \mathcal{B}(\mathcal{H})$ in the usual sense of von Neumann algebra tensor products.

Define the semifinite tensor product trace $\text{Tr}_\Gamma = \text{tr}_\Gamma \otimes \text{Tr}$ on each of the algebras $\mathcal{A}_{\mathcal{H}}^L(\Gamma, \bar{\sigma})$ and $\mathcal{A}_{\mathcal{H}}^R(\Gamma, \sigma)$. Here Tr denotes the standard (semi-finite) trace on $\mathcal{B}(\mathcal{H})$.

Let us recall that a closed linear subspace $V \subset \widetilde{\mathcal{H}}$ is *affiliated* to a von Neumann algebra \mathcal{A} of operators in a Hilbert space $\widetilde{\mathcal{H}}$ if $P_V \in \mathcal{A}$ where P_V is the orthogonal projection on V in $\widetilde{\mathcal{H}}$. This is equivalent to saying that V is invariant under the commutant \mathcal{A}' of \mathcal{A} in $\widetilde{\mathcal{H}}$.

The *von Neumann dimension* \dim_Γ of a closed subspace V of $\ell^2(\Gamma) \otimes \mathcal{H}$ that is invariant under $\{T_\gamma^L \otimes 1, \gamma \in \Gamma\}$ (or, equivalently, affiliated to $\mathcal{A}_{\mathcal{H}}^R(\Gamma, \sigma)$) is defined as

$$\dim_\Gamma(V) = \text{Tr}_\Gamma(P_V).$$

The same formula is used for the subspaces which are invariant under $\{T_\gamma^R \otimes 1, \gamma \in \Gamma\}$ (or, equivalently, affiliated to $\mathcal{A}_{\mathcal{H}}^L(\Gamma, \bar{\sigma})$).

Also the *von Neumann rank* of an operator $Q \in \mathcal{A}_{\mathcal{H}}^R(\Gamma, \sigma)$ or $Q \in \mathcal{A}_{\mathcal{H}}^L(\Gamma, \bar{\sigma})$ is defined as

$$\text{rank}_\Gamma(Q) = \dim_\Gamma(\overline{\text{Range}(Q)}),$$

where the bar over $\text{Range}(Q)$ means closure.

1.3. Magnetic Hamiltonians. The *magnetic Laplacian* on $L^2(\widetilde{M})$ is defined as

$$H_A = \nabla^* \nabla = (d + iA)^* (d + iA)$$

and more generally, the *magnetic Hamiltonian* or *Magnetic Schrödinger operator* is defined as

$$H_{A,V}(\mu) = \mu H_A + \mu^{-1} V,$$

where V is any Γ -invariant smooth function on \widetilde{M} . The Hamiltonian $H = H_{A,V}(\mu)$ is a self adjoint second order elliptic differential operator. It commutes with the magnetic translations T_γ (for all $\gamma \in \Gamma$), i.e. with the (Γ, σ) -action which was defined above. To see this note first that the operators $U_\gamma = (\gamma^{-1})^*$ and S_γ (the multiplication by $\exp(-i\psi_\gamma)$) are defined not only on scalar functions but also on 1-forms (and actually on p -forms for any $p \geq 0$) on \widetilde{M} . Hence the magnetic translations T_γ are well defined on forms as well. The operators T_γ are obviously unitary on the L^2 spaces of forms, where the L^2 structure is defined by the fixed Γ -invariant metric on \widetilde{M} . An easy calculation shows that $T_\gamma \nabla = \nabla T_\gamma$ on scalar functions. By taking adjoint operators we obtain $T_\gamma \nabla^* = \nabla^* T_\gamma$ on 1-forms. Therefore $T_\gamma H_A = H_A T_\gamma$ on functions. Since obviously $T_\gamma V = V T_\gamma$, we see that $H_{A,V}(\mu)$ commutes with T_γ for all γ .

In dimension 2, it is the spectrum and the spectral projections of the magnetic Schrödinger operator that are of fundamental importance to the study of the quantum Hall effect. We remark that it is virtually impossible to explicitly compute the spectrum of $H_{A,V}$ for arbitrary V which is Γ -invariant, even in 2 dimensions and for simplest manifolds. Nevertheless, Comtet and Houston [9] computed the spectrum of H_A (with $V = 0$) on the hyperbolic plane with the magnetic potential $A = \theta y^{-1} dx$ (which corresponds to the constant magnetic field), where we can assume

without loss of generality that $\theta > 0$. This spectrum is the union of a finite number of eigenvalues $\{(2k+1)\theta - k(k+1) : k = 0, 1, 2, \dots < \theta - \frac{1}{2}\}$ and the continuous spectrum $[\frac{1}{4} + \theta^2, \infty)$.

Since H commutes with the (Γ, σ) -action, it follows by the spectral mapping theorem that the spectral projections of H , $E_\lambda = \chi_{(-\infty, \lambda]}(H)$ are bounded operators on $L^2(\widetilde{M})$ that also commute with the (Γ, σ) -action i.e. $T_\gamma E_\lambda = E_\lambda T_\gamma$, $\forall \gamma \in \Gamma$. The *commutant* of the (Γ, σ) -action is a von Neumann algebra

$$\mathcal{U}_{\widetilde{M}}(\Gamma, \bar{\sigma}) = \left\{ Q \in \mathcal{B}(L^2(\widetilde{M})) : T_\gamma Q = QT_\gamma, \quad \forall \gamma \in \Gamma \right\}.$$

To characterize the Schwartz kernels $k_Q(x, y)$ of the operators $Q \in \mathcal{U}_{\widetilde{M}}(\Gamma, \bar{\sigma})$ note that the relation $T_\gamma Q = QT_\gamma$ can be rewritten in the form

$$e^{i\psi_\gamma(x)} k_Q(\gamma x, \gamma y) e^{-i\psi_\gamma(y)} = k_Q(x, y), \quad \forall x, y \in \widetilde{M} \quad \forall \gamma \in \Gamma,$$

so $Q \in \mathcal{U}_{\widetilde{M}}(\Gamma, \bar{\sigma})$ if and only if this holds for all $\gamma \in \Gamma$. In particular, in this case $k_Q(x, x)$ is Γ -invariant. For the spectral projections of H we also have $E_\lambda \in \mathcal{U}_{\widetilde{M}}(\Gamma, \bar{\sigma})$, so the corresponding Schwartz kernels also satisfy the relations above. Note that the Schwartz kernels of E_λ are in $C^\infty(\widetilde{M} \times \widetilde{M})$.

To define a natural trace on $\mathcal{U}_{\widetilde{M}}(\Gamma, \bar{\sigma})$ we will construct an isomorphism of this algebra with the von Neumann algebra $\mathcal{A}_{\mathcal{H}}^L(\Gamma, \bar{\sigma})$, where $\mathcal{H} = L^2(\mathcal{F})$.

Let \mathcal{F} be a fundamental domain for the Γ -action on \widetilde{M} . By choosing a connected fundamental domain \mathcal{F} for the action of Γ on \widetilde{M} , we can define a (Γ, σ) -equivariant isometry

$$(1.1) \quad \mathbf{U} : L^2(\widetilde{M}) \cong \ell^2(\Gamma) \otimes L^2(\mathcal{F})$$

as follows. Let $i : \mathcal{F} \rightarrow \widetilde{M}$ denote the inclusion map. Define

$$\mathbf{U}(\phi) = \sum_{\gamma \in \Gamma} \delta_\gamma \otimes i^*(T_\gamma \phi), \quad \phi \in L^2(\widetilde{M}).$$

Lemma 1.1. *The map $\mathbf{U} : L^2(\widetilde{M}) \rightarrow \ell^2(\Gamma) \otimes L^2(\mathcal{F})$ defined above is a (Γ, σ) -equivariant isometry, where the (Γ, σ) -action is given by the operators T_γ and $T_\gamma^R \otimes 1$ on the spaces $L^2(\widetilde{M})$ and $\ell^2(\Gamma) \otimes L^2(\mathcal{F})$ respectively.*

Proof. Given $\phi \in L^2(\widetilde{M})$, we compute

$$\begin{aligned} \mathbf{U}(T_\gamma \phi) &= \sum_{\gamma' \in \Gamma} \delta_{\gamma'} \otimes i^*(T_{\gamma'} T_\gamma \phi) = \sum_{\gamma' \in \Gamma} \sigma(\gamma', \gamma) \delta_{\gamma'} \otimes i^*(T_{\gamma' \gamma} \phi) \\ &= \sum_{\gamma' \in \Gamma} \sigma(\gamma' \gamma^{-1}, \gamma) \delta_{\gamma' \gamma^{-1}} \otimes i^*(T_{\gamma'} \phi) = (T_\gamma^R \otimes 1) \mathbf{U} \phi, \end{aligned}$$

which proves that \mathbf{U} is a (Γ, σ) -equivariant map. It is straightforward to check that that \mathbf{U} is an isometry. \square

Since $\mathcal{U}_{\widetilde{M}}(\Gamma, \bar{\sigma})$ is the commutant of $\{T_\gamma\}_{\gamma \in \Gamma}$, and $\mathcal{A}_{\mathcal{H}}^L(\Gamma, \bar{\sigma})$, with $\mathcal{H} = L^2(\mathcal{F})$, is the commutant of $\{T_\gamma^R \otimes 1\}_{\gamma \in \Gamma}$, we see that \mathbf{U} induces an isomorphism of von Neumann algebras $\mathcal{U}_{\widetilde{M}}(\Gamma, \bar{\sigma})$ and $\mathcal{A}_{\mathcal{H}}^L(\Gamma, \bar{\sigma})$. Therefore we can transfer the trace Tr_Γ from $\mathcal{A}_{\mathcal{H}}^L(\Gamma, \bar{\sigma})$ to $\mathcal{U}_{\widetilde{M}}(\Gamma, \bar{\sigma})$. The result will be a semifinite Γ -trace on $\mathcal{U}_{\widetilde{M}}(\Gamma, \bar{\sigma})$ which we will still denote Tr_Γ .

It is easy to check that for any $Q \in \mathcal{U}_{\widetilde{M}}(\Gamma, \bar{\sigma})$ with a finite Γ -trace and a continuous Schwartz kernel k_Q we have

$$\text{Tr}_\Gamma Q = \int_{\mathcal{F}} k_Q(x, x) dx$$

where dx means the Γ -invariant measure. An important particular case is a spectral projection E_λ of the magnetic Schrödinger operator $H = H_{A,V}$ as considered above. The projection E_λ has a finite Γ -trace and a C^∞ Schwartz kernel. Therefore we can define a *spectral density function*

$$N_\Gamma(\lambda; H) = \text{Tr}_\Gamma E_\lambda,$$

which is finite for all $\lambda \in \mathbb{R}$. It is easy to see that $\lambda \mapsto N_\Gamma(\lambda; H)$ is a non-decreasing function, and the spectrum of H can be reconstructed as the set of its points of growth, i.e.

$$\text{spec}(H) = \{\lambda \in \mathbb{R} : N_\Gamma(\lambda + \varepsilon; H) - N_\Gamma(\lambda - \varepsilon; H) > 0, \forall \varepsilon > 0\}.$$

The *von Neumann dimension* \dim_Γ of a closed subspace V of $L^2(\widetilde{M})$ is well defined if V is invariant under T_γ , $\forall \gamma \in \Gamma$. Also the *von Neumann rank* of an operator $Q \in \mathcal{U}_{\widetilde{M}}(\Gamma, \bar{\sigma})$ is well defined.

1.4. Gauge invariance. If we make another choice of vector potential A' such that $dA' = B$, then it follows that $A' - A$ is a closed 1-form on a simply connected manifold \widetilde{M} , and therefore it is exact, i.e. $A' = A + d\phi$, where $\phi \in C^\infty(\widetilde{M})$. We will always assume that ϕ is normalized by the condition $\phi(x_0) = 0$. It follows that ϕ is real-valued.

It then follows that the connection $\nabla = d + iA$ gets unitarily conjugated into a new connection $\nabla' = d + iA' = e^{-i\phi}\nabla e^{i\phi}$. Therefore $\nabla'^* = e^{-i\phi}\nabla^*e^{i\phi}$ and $H' = e^{-i\phi}He^{i\phi}$, where $H = H_{A,V}$, $H' = H_{A',V}$, and V is Γ -invariant. In particular, H' and H are unitarily equivalent. Let us repeat the constructions of the previous subsections with A replaced by A' indicating relations of the modified objects with the old ones.

Define the function ψ'_γ from $d\psi'_\gamma = \gamma^*A' - A'$, $\psi'_\gamma(x_0) = 0$ (with the same point $x_0 \in M$ as above). Then

$$\psi'_\gamma = \psi_\gamma + \gamma^*\phi - \phi - \phi(\gamma x_0).$$

The new cocycle will be

$$\sigma'(\gamma_1, \gamma_2) = \exp(-i\psi'_{\gamma_1}(\gamma_2 x_0)) = \sigma(\gamma_1, \gamma_2) \exp(-i[\phi(\gamma_1 \gamma_2 x_0) - \phi(\gamma_1 x_0) - \phi(\gamma_2 x_0)]).$$

The modified magnetic translations are defined by $T'_\gamma = U'_\gamma S'_\gamma$, where $U'_\gamma = U_\gamma = (\gamma^{-1})^*$ and $S'_\gamma = \exp(-i\psi'_\gamma)$. Then $T'_{\gamma_1} T'_{\gamma_2} = \sigma'(\gamma_1, \gamma_2) T'_{\gamma_1 \gamma_2}$.

Clearly H' commutes with the modified magnetic translations T'_γ , $\forall \gamma \in \Gamma$. The relation between old and new magnetic translations is

$$T'_\gamma = e^{i\phi(\gamma \cdot x_0)} \left(e^{-i\phi} T_\gamma e^{i\phi} \right),$$

which is again the same unitary conjugation up to a constant unitary factor.

Now we can introduce a von Neumann algebra

$$\mathcal{U}_{\widetilde{M}}(\Gamma, \bar{\sigma}') = \{Q' \in \mathcal{B}(L^2(\widetilde{M})) : T'_\gamma Q' = Q' T'_\gamma, \forall \gamma \in \Gamma\}.$$

Clearly the map

$$\mathcal{U}_{\widetilde{M}}(\Gamma, \bar{\sigma}) \longrightarrow \mathcal{U}_{\widetilde{M}}(\Gamma, \bar{\sigma}'), \quad Q \longmapsto Q' = e^{-i\phi} Q e^{i\phi},$$

is an isometric \star -isomorphism of von Neumann algebras. It is easy to see that this isomorphism preserves the Γ -trace which is defined on both algebras. If an operator Q has a smooth Schwartz kernel $k_Q(x, y)$ and a finite Γ -trace, then

$$k_{Q'}(x, y) = e^{-i\phi(x)} k_Q(x, y) e^{i\phi(y)}, \quad \forall x, y \in \widetilde{M},$$

hence

$$k_{Q'}(x, x) = k_Q(x, x), \quad \forall x \in \widetilde{M},$$

so the equality $\text{Tr}_\Gamma Q' = \text{Tr}_\Gamma Q$ follows from the expression of the Γ -traces in terms of kernels.

If E_λ and E'_λ are spectral projections of H and $H' = e^{-i\phi} H e^{i\phi}$ respectively, then clearly $E'_\lambda = e^{-i\phi} E_\lambda e^{i\phi}$, therefore $\text{Tr}_\Gamma E'_\lambda = \text{Tr}_\Gamma E_\lambda$. This means that the spectral density functions of H and H' coincide, i.e. the spectral density function is gauge invariant.

2. SEMICLASSICAL APPROXIMATION AND THE EXISTENCE OF SPECTRAL GAPS

We will study an L^2 -version of semiclassical approximation, which is similar to the ones which appear when we take the Witten deformation of the de Rham complex and consider the corresponding Laplacian (cf. [20, 10, 12]). For the case of the algebra corresponding to the regular representation of $\pi_1(M)$, such asymptotics were first proved in [19], see also [5] for a related semiclassical approximation technique. The proofs given in [19] will be adapted to work in the more general situation that we need in this section.

Recall that $H = H_{A,V}(\mu) = \mu H_A + \mu^{-1} V$ is a second order differential operator acting on $L^2(\widetilde{M})$, such that it commutes with the projective unitary (Γ, σ) -action on $L^2(\widetilde{M})$, given by the magnetic translations $\{T_\gamma, \gamma \in \Gamma\}$. Moreover, note that H is a second order elliptic operator with a positive principal symbol, order operator, V is a non-negative potential function on M which has only nondegenerate zeroes and $\mu > 0$ is a small parameter.

Actually the results will not change if we add to H any Γ -invariant zeroth order operator, i.e. multiplication by a smooth Γ -invariant function.

Let us recall that $N_\Gamma(\lambda; H)$ denote the von Neumann spectral density function of the operator H which can be defined as

$$N_\Gamma(\lambda, H) = \text{Tr}_\Gamma E_\lambda(H) = \text{Tr}_\Gamma (\chi_{(-\infty, \lambda]}(H)),$$

where χ_F means the characteristic function of a subset $F \subset \mathbb{R}$.

Let us choose a fundamental domain $\mathcal{F} \subset \widetilde{M}$ so that there is no zeros of V on the boundary of \mathcal{F} . This is equivalent to saying that the translations $\{\gamma\mathcal{F}, \gamma \in \Gamma\}$ cover the set $V^{-1}(0)$ (the set of all zeros of V). Let $V^{-1}(0) \cap \mathcal{F} = \{\bar{x}_j | j = 1, \dots, N\}$ be the set of all zeros of V in \mathcal{F} ; $\bar{x}_i \neq \bar{x}_j$ if $i \neq j$. Let K denote the *model operator* of H (cf. [19]), which is obtained as a direct sum of quadratic parts of H in all points $\bar{x}_1, \dots, \bar{x}_N$. More precisely,

$$K = \bigoplus_{1 \leq j \leq N} K_j,$$

where K_j is an unbounded self-adjoint operator in $L^2(\mathbb{R}^n)$ which corresponds to the zero \bar{x}_j . It is a quantum harmonic oscillator and has a discrete spectrum. We assume that we have fixed local coordinates on \widetilde{M} in a small neighborhood $B(\bar{x}_j, r)$ of \bar{x}_j for every $j = 1, \dots, N$. Then K_j has the form

$$K_j = H_j^{(2)} + V_j^{(2)},$$

where all the components are obtained from H as follows. In the fixed local coordinates on \widetilde{M} near \bar{x}_j , the second order term $H_j^{(2)}$ is a homogeneous second order differential operator with constant coefficients (without lower order terms) obtained by isolating the second order terms in the operator H and freezing the coefficients of this operator at \bar{x}_j . (Note that $H_j^{(2)}$ does not depend of A .) The zeroth order term $V_j^{(2)}$ is obtained by taking the quadratic part of V in the chosen coordinates near \bar{x}_j .

More explicitly,

$$H_j^{(2)} = \sum_{i,k=1}^n g^{ik}(\bar{x}_j) \frac{\partial^2}{\partial x_i \partial x_k}, \quad V_j^{(2)} = \frac{1}{2} \sum_{i,k=1}^n \frac{\partial^2 V}{\partial x_i \partial x_k}(\bar{x}_j) x_i x_k,$$

where (g^{ik}) is the inverse matrix to the matrix of the Riemannian tensor (g_{ik}) .

We will say that H is *flat* near \bar{x}_j if $H = K_j$ near \bar{x}_j . (In particular, in this case we should have $A = 0$ near \bar{x}_j .)

We will also need the operator

$$K(\mu) = \bigoplus_{1 \leq j \leq N} K_j(\mu),$$

where

$$K_j(\mu) = \mu H_j^{(2)} + \mu^{-1} V_j^{(2)}, \quad \mu > 0.$$

It is easy to see that $K(\mu)$ has the same spectrum as $K = K(1)$.

Let $\{\alpha_p : p \in \mathbb{N}\}$ denote the set of all eigenvalues of the model operator K , $\alpha_p \neq \alpha_q$ for $p \neq q$, and r_p denote the multiplicity of α_p , i.e. $r_p = \dim \text{Ker}(K - \alpha_p I)$ where K is considered in $(L^2(\mathbb{R}^n))^N$. Denote by $N(\lambda; K)$ the distribution function of the eigenvalues of K , i.e. $N(\lambda; K)$ is the number of eigenvalues which are $\leq \lambda$ (multiplicities counted).

The following is the main result of this paper.

Theorem 2.1 (Semiclassical Approximation). *For any $R > 0$ there exist constants $C > 0$ and $\mu_0 > 0$ such that for any $\mu \in (0, \mu_0)$ and $\lambda \leq R$, one has*

$$N(\lambda - C\mu^{1/5}; K) \leq N_\Gamma(\lambda; H) \leq N(\lambda + C\mu^{1/5}; K).$$

Therefore

$$\text{spec}(H) \cap (-\infty, R] \subset \bigcup_{p=1}^{\infty} (\alpha_p - C\mu^{1/5}, \alpha_p + C\mu^{1/5}),$$

where $\{\alpha_p : p \in \mathbb{N}\}$ denotes the set of all eigenvalues of the model operator K . Moreover for any $p = 1, 2, 3, \dots$ with $\alpha_p \in [-R, R]$ and any $\mu \in (0, \mu_0)$ one has

$$N_\Gamma(\alpha_p + C\mu^{1/5}; H) - N_\Gamma(\alpha_p - C\mu^{1/5}; H) = r_p = N(\alpha_p + 0; K) - N(\alpha_p - 0; K).$$

This means that for small values of μ , the spectrum of H concentrates near the eigenvalues of the model operator K , and for every such eigenvalue, the von Neumann dimension of the spectral subspace of the operator H , corresponding to the part of the spectra near the eigenvalue, is exactly equal to the usual multiplicity of this eigenvalue of K .

Proof of Theorem on Existence of Spectral Gaps. Let us assume that $\lambda \in \mathbb{R}$ and $\lambda \notin \text{spec}(K)$, and then choose $R > \lambda$. Let $C > 0$ and $\mu_0 > 0$ be as in Theorem 2.1. Then by taking even smaller μ_0 we will get

$$\lambda \notin \bigcup_{p=1}^{\infty} (\alpha_p - C\mu^{1/5}, \alpha_p + C\mu^{1/5}), \quad \forall \mu \in (0, \mu_0).$$

Therefore $\lambda \notin \text{spec}(H)$. □

The proof of Theorem 2.1 will be divided into 2 parts: estimating $N_\Gamma(\lambda, H)$ from below and from above.

2.1. Estimate from below. We will start by proving an estimate from below for $N_\Gamma(\lambda; H)$ where $H = H(\mu)$, $\lambda \leq R$ with an arbitrarily fixed $R > 0$, and $\mu \downarrow 0$.

Our main tool will be the standard variational principle for the spectral density function.

Lemma 2.2 (Variational principle). *For every $\lambda \in \mathbb{R}$*

$$(1) \quad N_\Gamma(\lambda; H) = \sup \{ \dim_\Gamma L \mid L \subset \text{Dom}(H); \quad (Hf, f) \leq \lambda(f, f), \quad \forall f \in L \}.$$

It is understood here that L should be a closed (Γ, σ) -invariant subspace in $L^2(\widetilde{M})$, i.e. closed subspace which is invariant under all magnetic translations T_γ , $\gamma \in \Gamma$.

A similar variational principle for the usual action of Γ was used e.g. in [11, 19].

We will now describe an appropriate construction of a test space L .

Fix a function $J \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq J \leq 1$, $J(x) = 1$ if $|x| \leq 1$, $J(x) = 0$ if $|x| \geq 2$, and $(1 - J^2)^{1/2} \in C^\infty(\mathbb{R}^n)$. Let us fix a number κ , $0 < \kappa < 1/2$, which we shall choose later. For any

$\mu > 0$ define $J^{(\mu)}(x) = J(\mu^{-\kappa}x)$. This will be our standard cut-off function. Let $J_j = J^{(\mu)}$ in the fixed coordinates near \bar{x}_j . Denote also $J_{j,\gamma} = (\gamma^{-1})^*J_j$. (This function is supported near $\gamma\bar{x}_j$.)

We will always take $\mu \in (0, \mu_0)$ where μ_0 is sufficiently small, so in particular the supports of all functions $J_{j,\gamma}$ are disjoint. Denote

$$J_0 = (1 - \sum_{j,\gamma} J_{j,\gamma}^2)^{1/2}.$$

Clearly, $J_0 \in C^\infty(\widetilde{M})$. Note that there exists $c_0 > 0$ such that $V \geq c_0\mu^{2\kappa}$ on $\text{supp } J_0$.

functions $J_{j,\gamma}$ operators.

Denote by $\{\psi_{m,j} \mid m = 1, 2, \dots\}$ an orthonormal system of eigenfunctions of the operator K_j in $L^2(\mathbb{R}^n)$ where the coordinates in a neighborhood of the origin $0 \in \mathbb{R}^n$ are identified with the chosen coordinates near \bar{x}_j , so that 0 corresponds to \bar{x}_j . The corresponding eigenvalues will be denoted $\lambda_{m,j}$.

Let us define $\phi_{m,j} = J_j \psi_{m,j}$, extended by 0 outside of a fixed small ball centered at \bar{x}_j . Then $\phi_{m,j} \in C_0^\infty(\widetilde{M})$ and it is supported near \bar{x}_j .

Lemma 2.3. *If $1/3 < \kappa < 1/2$, then the functions $\phi_{m,j}$ satisfy the following “almost orthogonality” relations*

$$(\phi_{m,j}, \phi_{m',j'}) = \delta_{j,j'}(\delta_{m,m'} + O(\mu^\kappa)),$$

$$(H\phi_{m,j}, \phi_{m',j'}) = \delta_{j,j'}(\lambda_{m,j}\delta_{m,m'} + O(\mu^{3\kappa-1})),$$

where $j = 1, \dots, N$, and m belongs to a finite set.

The proof is the same as the proof of Lemma 2.3 in [19]. The unboundedness of A does not matter because only a finite number of points \bar{x}_j is involved.

Not define $\phi_{m,j,\gamma} = T_\gamma \phi_{m,j}$. Then $\phi_{m,j,\gamma}$ is supported near $\gamma\bar{x}_j$.

Lemma 2.4. *If $1/3 < \kappa < 1/2$, then the functions $\phi_{m,j}$ satisfy the following “almost orthogonality” relations*

$$(\phi_{m,j,\gamma}, \phi_{m',j',\gamma'}) = \delta_{\gamma,\gamma'}\delta_{j,j'}(\delta_{m,m'} + O(\mu^\kappa)),$$

$$(H\phi_{m,j,\gamma}, \phi_{m',j',\gamma'}) = \delta_{\gamma,\gamma'}\delta_{j,j'}(\lambda_{m,j}\delta_{m,m'} + O(\mu^{3\kappa-1})),$$

where $j = 1, \dots, N$, $\gamma \in \Gamma$, and m belongs to a finite set.

Proof. The first relation is obvious because the operator T_γ is unitary and moves supports by the action of γ . To prove the second estimate note that it is obvious if $\gamma \neq \gamma'$. If $\gamma = \gamma'$, then we get

$$(H\phi_{m,j,\gamma}, \phi_{m',j',\gamma}) = (HT_\gamma \phi_{m,j}, T_\gamma \phi_{m',j'}) = (T_\gamma^{-1} HT_\gamma \phi_{m,j}, \phi_{m',j'}) = (H\phi_{m,j}, \phi_{m',j'}),$$

and we can use the previous Lemma. \square

Now we will define two closed linear subspaces in $\Phi_\lambda^{\mathcal{F}} \subset L^2(\mathcal{F})$ and $\Phi_\lambda \subset L^2(\widetilde{M})$ as follows:

$$\Phi_\lambda^{\mathcal{F}} = \text{span}\{\phi_{m,j} \mid \lambda_{m,j} \leq \lambda\},$$

$$\Phi_\lambda = \text{span}^c\{\phi_{m,j,\gamma} \mid \lambda_{m,j} \leq \lambda\},$$

where span^c stands for closed linear span. Clearly $\dim \Phi_\lambda^{\mathcal{F}} < \infty$, and Φ_λ is a closed (Γ, σ) -invariant subspace in $L^2(\widetilde{M})$.

Lemma 2.5. $\dim \Phi_\lambda^{\mathcal{F}} = \dim_{\Gamma} \Phi_\lambda = N(\lambda; K)$.

Proof. Clearly $\Phi_\lambda^{\mathcal{F}} = \bigoplus_{j=1}^N \Phi_{\lambda,j}^{\mathcal{F}}$, where $\Phi_{\lambda,j}^{\mathcal{F}}$ is spanned by $\{\phi_{m,j}\}$ with fixed j . But we have $\dim \Phi_{\lambda,j}^{\mathcal{F}} = N(\lambda; K_j)$. Indeed, the eigenfunctions $\{\psi_{m,j} \mid m = 1, \dots, N(\lambda; K_j)\}$ are linearly independent and real analytic, so the corresponding $\phi_{m,j}$ are also linearly independent because $\phi_{m,j} = \psi_{m,j}$ near \bar{x}_j . It follows that $\dim \Phi_\lambda^{\mathcal{F}} = N(\lambda; K)$.

For any fixed j denote by $\{\tilde{\phi}_{m,j} \mid m = 1, \dots, N(\lambda; K_j)\}$ the orthonormal system which is obtained from the system $\{\phi_{m,j} \mid m = 1, \dots, N(\lambda; K_j)\}$ by the Gram-Schmidt orthogonalization process. Then $\{\tilde{\phi}_{m,j} \mid \lambda_{m,j} \leq \lambda\}$ is an orthonormal basis in $\Phi_\lambda^{\mathcal{F}}$, and $\{T_\gamma \tilde{\phi}_{m,j} \mid \lambda_{m,j} \leq \lambda, \gamma \in \Gamma\}$ is an orthonormal basis in Φ_λ . Note that $T_\gamma \tilde{\phi}_{m,j} \in C_0^\infty(\widetilde{M})$ and it is supported near $\gamma \bar{x}_j$.

Denote by $P_\lambda^{\mathcal{F}}$ and P_λ the orthogonal projections on $\Phi_\lambda^{\mathcal{F}}$ and Φ_λ respectively, $K_\lambda^{\mathcal{F}}$ and K_λ their Schwartz kernels. Then

$$K_\lambda^{\mathcal{F}}(x, y) = \sum_{\{m,j \mid \lambda_{m,j} \leq \lambda\}} \tilde{\phi}_{m,j} \otimes \overline{\tilde{\phi}_{m,j}} = \sum_{\{m,j \mid \lambda_{m,j} \leq \lambda\}} \tilde{\phi}_{m,j}(x) \overline{\tilde{\phi}_{m,j}(y)}.$$

In particular,

$$K_\lambda^{\mathcal{F}}(x, x) = \sum_{\{m,j \mid \lambda_{m,j} \leq \lambda\}} |\tilde{\phi}_{m,j}(x)|^2.$$

Similarly we find

$$K_\lambda(x, y) = \sum_{\{m,j,\gamma \mid \lambda_{m,j} \leq \lambda\}} T_\gamma \tilde{\phi}_{m,j} \otimes \overline{T_\gamma \tilde{\phi}_{m,j}}$$

and

$$K_\lambda(x, x) = \sum_{\{m,j,\gamma \mid \lambda_{m,j} \leq \lambda\}} |\tilde{\phi}_{m,j}(\gamma x)|^2.$$

It follows that $K_\lambda(x, x) = K_\lambda^{\mathcal{F}}(x, x)$ for all $x \in \mathcal{F}$, therefore

$$\dim_{\Gamma} \Phi_\lambda = \text{Tr}_{\Gamma}(P_\lambda) = \int_{\mathcal{F}} K_\lambda(x, x) dx = \int_{\mathcal{F}} K_\lambda^{\mathcal{F}}(x, x) dx = \text{Tr}(P_\lambda^{\mathcal{F}}) = \dim \Phi_\lambda^{\mathcal{F}} = N(\lambda; K),$$

which proves the lemma. \square

Proposition 2.6. For any $R > 0$ and $\kappa \in (0, 1/2)$ there exist $\mu_0 > 0$ and $C > 0$ such that for any $\lambda \leq R$ and any $\mu \in (0, \mu_0)$

$$(2) \quad N_{\Gamma}(\lambda + C\mu^\kappa; H) \geq N(\lambda; K).$$

Proof. Note first that if the estimate (2) holds with some $\kappa > 0$, it holds also for all smaller values of κ . Now we should argue as in the proof of Lemma 2.6 in [19] to conclude that $\Phi_\lambda \in \text{Dom}(H)$ and

$$(Hf, f) \leq (\lambda + C\mu^{3\kappa-1})(f, f), \quad f \in \Phi_\lambda.$$

Applying the variational principle (Lemma 2.2) we conclude that the estimate (2) holds with $3\kappa-1$ instead of κ . It remains to notice that $3\kappa-1$ takes all values from $(0, 1/2)$ when $1/3 < \kappa < 1/2$. \square

Corollary 2.7. *For any $R > 0$ and $\kappa \in (0, 1/2)$ there exist $\mu_0 > 0$ and $C > 0$ such that for any $\lambda \leq R$ and any $\mu \in (0, \mu_0)$*

$$(3) \quad N_\Gamma(\lambda; H) \geq N(\lambda - C\mu^\kappa; K).$$

Remark 2.8. If H is flat near each of the points \bar{x}_j , $j = 1, \dots, N$, then a better estimate is possible. Namely, for any $R > 0$, $\kappa \in (0, 1/2)$ and $\varepsilon > 0$ there exist $\mu_0 > 0$ and $C > 0$ such that for any $\lambda \leq R$ and any $\mu \in (0, \mu_0)$

$$(4) \quad N_\Gamma(\lambda; H) \geq N(\lambda - C \exp(-C^{-1}\mu^{-1+\varepsilon}); K).$$

(Arguments in Sect.2 of [19] apply here too.)

2.2. Estimate from above. Here we will prove an estimate from above for $N_\Gamma(\lambda; H)$, similar to the estimate (4).

Lemma 2.9. *Assume that there exists an operator $D \in \mathcal{U}_{\widetilde{M}}(\Gamma, \bar{\sigma})$ (i.e. a bounded operator in $L^2(\widetilde{M})$ commuting with all magnetic translations T_γ , $\gamma \in \Gamma$), such that*

$$(a) \text{rank}_\Gamma D \leq \tilde{k};$$

$$(b) H + D \geq \tilde{\lambda}.$$

Then $N_\Gamma(\tilde{\lambda} - \varepsilon; H) \leq \tilde{k}$ for any $\varepsilon > 0$.

The proof does not differ from the proof of Lemma 3.7 in [19].

We will prove that for any $R > 0$ there exist $C > 0$ and $\mu_0 > 0$ such that for any $\lambda < R$ and $\mu \in (0, \mu_0)$ there exists an operator $D \in \mathcal{U}_{\widetilde{M}}(\Gamma, \bar{\sigma})$, which satisfies the conditions (a), (b) with $\tilde{k} = N(\lambda; K)$, $\tilde{\lambda} = \lambda - C\mu^{1/5}$. It would follow from Lemma 2.9 that

$$(5) \quad N_\Gamma(\lambda - C\mu^{1/5}; H) \leq N(\lambda; K),$$

hence

$$(6) \quad N_\Gamma(\lambda; H) \leq N(\lambda + C\mu^{1/5}; K),$$

which is the desired estimate from above.

Denote $E_\lambda^{(j)}(\mu) = \chi_{(-\infty, \lambda]}(K_j(\mu))$ which is the spectral projection of $K_j(\mu)$. It is an operator of the finite rank $N(\lambda; K_j)$ in $L^2(\mathbb{R}^n)$. Identifying the coordinates in a neighborhood of $0 \in \mathbb{R}^n$ with the chosen coordinates near \bar{x}_j , so that $0 \in \mathbb{R}^n$ corresponds to \bar{x}_j , we can form an operator

$D_j = LJ_j E_\lambda^{(j)}(\mu) J_j$ in $L^2(\widetilde{M})$, where $L \in \mathbb{R}$, $L > \lambda$. Clearly D_j has a smooth Schwartz kernel supported in a neighborhood of (\bar{x}_j, \bar{x}_j) in $\widetilde{M} \times \widetilde{M}$.

Now denote

$$\mathcal{D}_{\mathcal{F}} = \sum_{j=1}^n D_j$$

and

$$D = \sum_{\gamma \in \Gamma} T_\gamma \mathcal{D}_{\mathcal{F}} T_\gamma^{-1}.$$

It is easy to check that $T_\gamma D = DT_\gamma$, $\forall \gamma \in \Gamma$.

Note that $\text{rank } E_\lambda^{(j)}(\mu) = N(\lambda; K_j)$, hence $\text{rank } D_j \leq N(\lambda; K_j)$ and $\text{rank } \mathcal{D}_{\mathcal{F}} \leq N(\lambda; K)$. (In fact, it is easy to see that we have equalities in both inequalities above, but we do not need this.) Also $\mathcal{D}_{\mathcal{F}}$ has a Schwartz kernel in $C_0^\infty(\mathcal{F} \times \mathcal{F})$.

Denote by $P_{\mathcal{F}}$ the orthogonal projection on the image of $\mathcal{D}_{\mathcal{F}}$. Choosing an orthonormal basis in this image we see that the Schwartz kernel of $P_{\mathcal{F}}$ is also in $C_0^\infty(\mathcal{F} \times \mathcal{F})$. For the orthogonal projection P on the closure of the image of D we have

$$P = \sum_{\gamma \in \Gamma} T_\gamma P_{\mathcal{F}} T_\gamma^{-1}.$$

Clearly $\text{rank } D_{\mathcal{F}} = \text{Tr } P_{\mathcal{F}}$ and $\text{rank}_{\Gamma} D = \text{Tr}_{\Gamma} P$. But the argument from the proof of Lemma 2.5 of the previous subsection shows that $\text{Tr}_{\Gamma} P = \text{Tr } P_{\mathcal{F}}$, so

$$\text{rank}_{\Gamma} D = \text{rank } D_{\mathcal{F}} \leq N(\lambda; K),$$

which proves the condition (a) with $\tilde{k} = N(\lambda; K)$.

To verify the condition (b) we will use the IMS localization technique adopting the manifold version explained in [19].

Using the same functions $J_0, J_{j,\gamma}$ as in the previous section, we have on \widetilde{M} :

$$J_0^2 + \sum_{j,\gamma} J_{j,\gamma}^2 = 1.$$

We will also identify the functions $J_0, J_{j,\gamma}$ with the corresponding multiplication operators.

Lemma 2.10 (The IMS localization formula). *The following operator identity is true:*

$$\begin{aligned} (\text{L}) \quad H &= J_0 H J_0 + \sum_{j,\gamma} J_{j,\gamma} H J_{j,\gamma} + \frac{1}{2}[J_0, [J_0, H]] + \frac{1}{2} \sum_{j,\gamma} [J_{j,\gamma}, [J_{j,\gamma}, H]] \\ &= J_0 H J_0 + \sum_{j,\gamma} J_{j,\gamma} H J_{j,\gamma} - \mu a^{(2)}(x, dJ_0(x)) - \mu \sum_{j,\gamma} a^{(2)}(x, dJ_{j,\gamma}(x)), \end{aligned}$$

where $a^{(2)}$ is the principal symbol of H , considered as a function on $T^* \widetilde{M}$.

The proof can be found e.g. in [19].

The last two terms in (L) are Γ -invariant and easily estimated as $O(\mu^{1-2\kappa})$, so we have

$$(7) \quad H \geq J_0 H J_0 + \sum_{j,\gamma} J_{j,\gamma} H J_{j,\gamma} - C\mu^{1-2\kappa} I.$$

Since $\mu H_A \geq 0$, we also have with some constant $c > 0$

$$(8) \quad J_0 H J_0 \geq \mu^{-1} V J_0^2 \geq c\mu^{-1+2\kappa} J_0^2,$$

if $\mu \in (0, \mu_0)$, and μ_0 is sufficiently small. Note that the coefficient in the right hand side here tends to $+\infty$ as $\mu \downarrow 0$.

Following Lemma 3.4 in [19] we find that

$$(9) \quad J_j H J_j \geq (1 - C\mu^\kappa) J_j K_j(\mu) J_j - C\mu^{3\kappa-1} J_j^2.$$

But we also have

$$\begin{aligned} J_{j,\gamma} H J_{j,\gamma} &= J_{j,\gamma} T_\gamma H T_\gamma^{-1} J_{j,\gamma} = T_\gamma J_j H J_j T_\gamma^{-1} \\ &\geq (1 - C\mu^\kappa) T_\gamma J_j K_j(\mu) J_j T_\gamma^{-1} - C\mu^{3\kappa-1} T_\gamma J_j^2 T_\gamma^{-1} \\ &= (1 - C\mu^\kappa) T_\gamma J_j K_j(\mu) J_j T_\gamma^{-1} - C\mu^{3\kappa-1} J_{j,\gamma}^2. \end{aligned}$$

Let us sum over j, γ and add D and also $J_0 H J_0$. Using the inequality

$$K_j(\mu) + L E_\lambda^{(j)} \geq \lambda I,$$

and also the estimates (7),(8),(9) above, we obtain then

$$H + D \geq c\mu^{-1+2\kappa} J_0^2 + (1 - C\mu^\kappa) \lambda \sum_{j,\gamma} J_{j,\gamma}^2 - C\mu^{3\kappa-1} \sum_{j,\gamma} J_{j,\gamma}^2 - C\mu^{1-2\kappa} I.$$

Choosing here $\kappa = 2/5$ we obtain

$$(10) \quad H + D \geq (\lambda - C\mu^{1/5}) I$$

with a constant C and with $\mu \in (0, \mu_0)$ for a sufficiently small μ_0 . This proves condition (b) and ends the proof of Theorem 2.1. \square

Remark 2.11. If H is flat near all points \bar{x}_j , then the estimate (6) can be improved as follows: for any $R > 0$ and $\varepsilon > 0$ there exist $C > 0$ and $\mu_0 > 0$ such that for all $\lambda < R$ and $\mu \in (0, \mu_0)$

$$N_\Gamma(\lambda; H) \leq N(\lambda + \mu^{1-\varepsilon}; K).$$

Together with the improved estimate from below this provides the inclusion

$$\text{spec}(H) \cap (-\infty, R] \subset \bigcup_{p=1}^{\infty} (\alpha_p - C\mu^{1-\varepsilon}, \alpha_p + C \exp(-C^{-1}\mu^{-1+\varepsilon})).$$

The necessary arguments can be found in [19].

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